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The D-bar method, inversion of certain integrals and integrability in 4 + 2 and 3 + 1 dimensions

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Abstract

We first review a method for deriving linear and nonlinear transform pairs, which is based on the spectral analysis of an eigenvalue equation and on the formulation of a *d*-bar problem. Then, we present two applications of this method: (a) we derive a certain linear transform pair in one dimension, which appears in the characterization of the Dirichlet-to-Neumann map of the Laplace equation in the interior of a convex two-dimensional curvilinear domain. (b) We derive a nonlinear Fourier transform pair in four dimensions, which can be used for the solution of the Cauchy problem of an integrable generalization of the Kadomtsev–Petviashvilli equation in 4 + 2, i.e. in four spatial and two temporal dimensions. The question of reducing this equation form 4 + 2 to 3 + 1 dimensions is also discussed.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

There exists a distinctive class of nonlinear equations called *integrable* [1]. The modern history of integrable equations begins with the celebrated works of Martin Kruskal and his colleagues [2] on the Cauchy problem of the Korteweg–deVries equation using, what was later called, the *inverse scattering transform* method. The next important step was taken by Peter Lax who established in [3] that the crucial property of an integrable equation is its formulation as the *compatibility of two linear eigenvalue equations*, which were later called a *Lax pair*.

The inverse scattering transform method for the solution of the Cauchy problem of integrable evolution equations in 1 + 1, i.e. in one spatial and one temporal dimensions, can be considered as a nonlinear Fourier transform method. The nonlinear analog of the relevant

Fourier transform pair can be constructed by performing the *spectral analysis* of the time-independent part of the Lax pair (see section 2) and by formulating a *Riemann–Hilbert problem* [4].

There *do* exist integrable nonlinear evolution equations in $2+1$, i.e. in *two* spatial and one temporal dimensions. For instance, a $2 + 1$ physically significant integrable generalization of the Korteweg–deVries equation is the Kadomtsev–Petviashvilli equation. A formal method for the solution of the Cauchy problem of equations in $2+1$ was developed by Mark Ablowitz, the author and their collaborators (this method was made rigorous in [5–7]). For equations in $2 + 1$, instead of a Riemann–Hilbert problem, one must now formulate either a *nonlocal Riemann–Hilbert* problem [8] or a *d-bar* problem [9] (the latter problem was first introduced in the field of integrability in the elegant analysis of Beals and Coifman [10] of certain problems in $1 + 1$, although for such problems a Riemann–Hilbert formalism is still adequate).

1.1. Inversion of integrals

The solution of the Cauchy problem of the Davey–Stewartson equation, which is a $2 + 1$ physically significant integrable generalization of the celebrated nonlinear Schrödinger equation, is based on the spectral analysis of the following eigenvalue equation [11–13]:

$$\frac{\partial\mu}{\partial x_1} + \mathrm{i}\sigma_3\frac{\partial\mu}{\partial x_2} - k[\sigma_3,\mu] = Q\mu, \qquad k\in\mathbb{C}, \qquad (x_1,x_2)\in\mathbb{R}^2, \tag{1.1}$$

where the eigenfunction $\mu(x_1, x_2, k, \bar{k})$ is a 2×2-matrix valued function, the bar denotes complex conjugation and

$$\sigma_3 = \mathrm{diag}(1,-1), \qquad Q(x_1,x_2) = \begin{pmatrix} 0 & q(x_1,x_2) \\ \bar{q}(x_1,x_2) & 0 \end{pmatrix}. \tag{1.2}$$

The spectral analysis of equation (1.1) yields a nonlinear Fourier transform pair in two spatial dimensions. Therefore, in the limit of small q, the formalism associated with (1.1) must reduce to a formalism for deriving the usual two-dimensional Fourier transform. Indeed, it was shown in [14] that if q is small then equation (1.1) reduces to the equation

$$\frac{\partial\mu}{\partial x_1} + \mathrm{i}\frac{\partial\mu}{\partial x_2} - k\mu = q, \qquad k\in\mathbb{C}, \qquad (x_1,x_2)\in\mathbb{R}^2 \tag{1.3}$$

and that the spectral analysis of this equation provides a novel derivation of the two-dimensional Fourier transform pair. This derivation can be considered as the construction of $q(x_1, x_2)$ in terms of its Fourier transform. Thus, the formalism introduced in [14] provides a novel method for *inventing a large class of integrals*. The first significant application of this method was the inversion by R Novikov [15] (see also [16]) of the so-called attenuated Radon transform. The next novel application of this method was the inversion by B Pelloni and the author [17] of the integral

$$\hat{f}(k) = \int_0^T \mathrm{e}^{-\mathrm{i}k^2t - \mathrm{i}kl(t)} f(t)\,\mathrm{d}t, \qquad k\in\mathbb{C}, \tag{1.4}$$

where $l(t)$ is a smooth function and T is a finite positive constant. This integral characterizes the Dirichlet-to-Neumann map for the linear version of the nonlinear Schrödinger equation, formulated in the moving domain $\{l(t) < x < \infty, 0 < t < T\}$ (the analogous problem for the heat equation is solved in [18]).

1.2. Integrability in $4 + 2$

One of the most important open questions in the field of integrability has been the question of the existence of integrable evolution equations in higher than two spatial dimensions. Substantial progress in this question was reported recently in [19] and [20] where it was shown respectively that (i) there exist integrable nonlinear evolution equations in *any* number of dimensions. However, these equations have the disadvantage that they involve a *nonlocal* commutator. (ii) There exist integrable nonlinear evolution equations in $4 + 2$, i.e. in *four* spatial and *two* temporal dimensions. In particular $4 + 2$ generalizations of the Kadomtsev–Petviashvilli and of the Davey–Stewartson equations were presented in [20]. Furthermore, the solution of the Cauchy problem of the latter equation was also presented in [20]. The question of reducing these equations from $4 + 2$ to $3 + 1$ dimensions was discussed in [21] and [22].

In the present paper (a) the steps needed for the spectral analysis of a given eigenvalue equation are reviewed in section 2 and illustrated with the aid of the eigenvalue equation associated with the Radon transform pair. (b) The spectral analysis of a certain eigenvalue equation in four dimensions is used in proposition 3.1 for the derivation of a certain nonlinear Fourier transform pair in four dimensions. This pair is then used for the solution of the Cauchy problem of a generalization of the Kadomtsev–Petviashvilli equation in $4+2$. (c) The spectral analysis of a certain eigenvalue equation in one dimension is used in proposition 4.1 for the inversion of the integral

$$\hat{f}(k) = \int_0^X \mathrm{e}^{-\mathrm{i}kx-kl(x)} f(x)\,\mathrm{d}x, \qquad k \in \mathbb{C},$$

where X is a positive constant, $l(x)$ is a given smooth function and $f(x)$ is an arbitrary function with appropriate smoothness. The above integral appears in the characterization of the Dirichlet-to-Neumann map of the Laplace equation in a domain involving the curve $y = l(x)$. The above results are further discussed in section 5.

2. The spectral analysis of an eigenvalue equation

In this section, starting from a given eigenvalue equation we review the main ideas and techniques needed for the construction of the associated transform pair $\{f, \hat{f}\}$. The relevant analysis, which will be referred to as the *spectral analysis*, involves two main steps: (i) solve the given eigenvalue equation in terms of f. If k denotes the eigenvalue parameter, this involves constructing a solution μ of the given eigenvalue equation which is *bounded* for *all* complex values of k. This problem will be referred to as the *direct problem*. (ii) Using the fact that μ is bounded for all complex k, construct an alternative representation of μ which (instead of depending on f) depends on some 'spectral function' of f denoted by $\hat{f}$. This problem will be referred to as the *inverse problem*.

It turns out that the inverse problem gives rise to either a Riemann–Hilbert or a d-bar problem. Indeed, for certain eigenvalue problems the function μ is *sectionally analytic* in k, i.e. it has different representations in different domains of the complex k-plane and each of these representations is analytic. In this case, if the 'jumps' of these representations across the different domains can be expressed in terms of $\hat{f}$, then it is possible to reconstruct μ as the solution of a Riemann–Hilbert problem which is uniquely defined in terms of $\hat{f}$. However, for a large class of eigenvalue problems, there exists a domain in the complex k-plane where μ is *not* analytic. In this case, if $\partial\mu/\partial\bar{k}$ can be expressed in terms of $\hat{f}$, then μ can be reconstructed through the solution of a d-bar problem which is uniquely defined in terms of $\hat{f}$.

We recall that the classical derivation of transform pairs involves the integration in the complex k-plane of an appropriate Green's function. However, this derivation is based on the assumption that the Green's function is an analytic function of k and it also assumes completeness. The assumption of analyticity corresponds to the case that μ is sectionally analytic. Therefore, the approach reviewed here has the advantage that, not only it provides a simpler approach to deriving classical transforms avoiding the problem of completeness, but also it can be applied to problems that the associated Green function is *not* an analytic function of k.

Example (The Radon transform). The Radon transform can be derived through the spectral analysis of the following eigenvalue equation for the scalar function $\mu(x_1, x_2, k)$:

$$\frac{1}{2}\left(k+\frac{1}{k}\right)\frac{\partial\mu}{\partial x_1}+\frac{1}{2\mathrm{i}}\left(k-\frac{1}{k}\right)\frac{\partial\mu}{\partial x_2}=f(x_1,x_2), \qquad k\in\mathbb{C}, \qquad (x_1,x_2)\in\mathbb{R}^2, \tag{2.1}$$

where f is an arbitrary function with appropriate smoothness and decay.

The solution of the direct problem involves constructing a function μ which is bounded for *all* complex values of k. It is shown in [16] (see also [23]) that such a solution is given by

$$\mu(x_1,x_2,k)=\frac{1}{2\pi\mathrm{i}}\operatorname{sgn}\left(\frac{1}{|k|^2}-|k|^2\right)\iint_{\mathbb{R}^2} f(x_1',x_2')\frac{\mathrm{d}x_1'\mathrm{d}x_2'}{z'-z}, \qquad |k|\neq 1, \qquad (x_1,x_2)\in\mathbb{R}^2, \tag{2.2}$$

where z is defined by

$$z=\frac{1}{2\mathrm{i}}\left(k-\frac{1}{k}\right)x_1-\frac{1}{2}\left(k+\frac{1}{k}\right)x_2. \tag{2.3}$$

The function μ defined by equation (2.2) is a sectionally analytic function of k for all k including $k=\infty$, since

$$\mu=O\left(\frac{1}{k}\right), \qquad k\to\infty. \tag{2.4}$$

The solution of the inverse problem involves constructing a representation for μ in terms of some spectral function $\hat{f}$. In this respect we note that the function μ defined by equation (2.2) has a 'jump' across the unit circle of the complex k-plane. Let μ^+ and μ^- denote the limits of μ as k approaches the unit circle from inside and outside the unit disc respectively, i.e.

$$\mu^{\pm}(x_1,x_2,\theta)\doteqdot\lim_{\varepsilon\to 0,\varepsilon>0}\mu(x_1,x_2,(1\mp\varepsilon)\,\mathrm{e}^{\mathrm{i}\theta}), \quad \theta\in[0,2\pi), \quad (x_1,x_2)\in\mathbb{R}^2. \tag{2.5}$$

The limits $\mu^{\pm}$ can be computed by analyzing equation (2.2). For this computation it is convenient to introduce the local coordinates, see figure 1,

$$x_1=\tau\cos\theta-\rho\sin\theta, \qquad x_2=\tau\sin\theta+\rho\cos\theta, \qquad (\tau,\rho)\in\mathbb{R}^2. \tag{2.6}$$

Using these coordinates, it is shown in [16] that the 'jump' of μ across the unit circle is given by

$$\mu^+(x_1,x_2,\theta)-\mu^-(x_1,x_2,\theta)=(H\hat{f})(\rho,\theta), \qquad \rho\in\mathbb{R}, \qquad \theta\in[0,2\pi), \tag{2.7}$$

where ρ can be expressed in terms of (x_1, x_2, θ) by the inverse of equations (2.6), i.e. by the equation

$$\rho=-x_1\sin\theta+x_2\cos\theta, \tag{2.8}$$

H denotes the Hilbert transform in the variable ρ and $\hat{f}(\rho,\theta)$ denotes the *Radon transform* of f, namely

$$\hat{f}(\rho,\theta)=\int_{-\infty}^{\infty} f(\tau\cos\theta-\rho\sin\theta,\tau\sin\theta+\rho\cos\theta)\,\mathrm{d}\tau, \qquad \rho\in\mathbb{R}, \qquad \theta\in[0,2\pi). \tag{2.9}$$

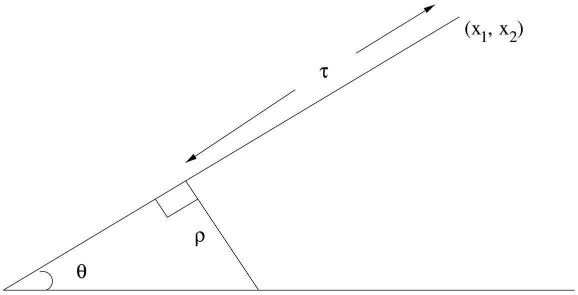

Figure 1. Local coordinates for the Radon transform.

Equations (2.4) and (2.7) define a Riemann–Hilbert problem for the sectionally analytic function μ. The unique solution of this problem for all $(x_1, x_2) \in \mathbb{R}^2$ is given by

$$\mu(x_1, x_2, k) = \frac{1}{2\mathrm{i}\pi} \int_{|k'|=1} (\mu^+ - \mu^-) \frac{\mathrm{d}k'}{k' - k}, \qquad k \in \mathbb{C}, \qquad |k| \neq 1.$$

Hence,

$$\mu(x_1, x_2, k) = \frac{1}{2\mathrm{i}\pi^2} \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta} - k} (H\hat{f})(\rho, \theta)\,\mathrm{d}\theta, \qquad k \in \mathbb{C}, \qquad |k| \neq 1, \tag{2.10}$$

where ρ is expressed in terms of (x_1, x_2, θ) by equation (2.8).

Equations (2.2) and (2.10) express μ in terms of f and $\hat{f}$ respectively. Using these two different representations for μ it is elementary to express f in terms of $\hat{f}$: replacing in equation (2.1) μ by the rhs of equation (2.10) we find the *inverse Radon transform*,

$$f(x_1, x_2) = \frac{1}{4\pi} (\partial_{x_1} - \mathrm{i}\partial_{x_2}) \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\theta} F(-x_1 \sin\theta + x_2 \cos\theta, \theta)\,\mathrm{d}\theta, \tag{2.11a}$$

where

$$F(\rho, \theta) = \frac{1}{\mathrm{i}\pi} ⨍_{-\infty}^{\infty} \frac{\hat{f}(\rho', \theta)}{\rho' - \rho}\,\mathrm{d}\rho' \tag{2.11b}$$

and $⨍$ denotes the principal value integral.

Equations (2.9) and (2.11) constitute the Radon transform pair.

Remark 2.1. It is shown in [16] that a slight variation of the above analysis yields the inversion of the attenuated Radon transform. The Radon transform provides the mathematical foundation of computerized tomography as well as of positron emission tomography, whereas the attenuated Radon transform provides the mathematical foundation of single photon emission computerized tomography (SPECT) [24]. The latter technique has important applications in many areas of medicine including oncology, cardiology and neurology. The numerical implementation of the inverse attenuated Radon transform using either cubic splines or Chebysev approximations is presented in [16] and [25]. A typical numerical implementation using a technique based on the fast Fourier transform is shown in the image (c) of figure 2. Figures 2(*b*)–(*d*) depict the numerical reconstructions of the realistic cardiac phantom depicted in figure 2(*a*), using three different techniques. The reconstruction (b) uses the approximation of zero attenuation which reduces the attenuated Radon transform to the classical Radon transform; the reconstruction of the later transform uses a technique based on the fast Fourier transform, which is called filter back projection (this is actually what is used now for SPECT

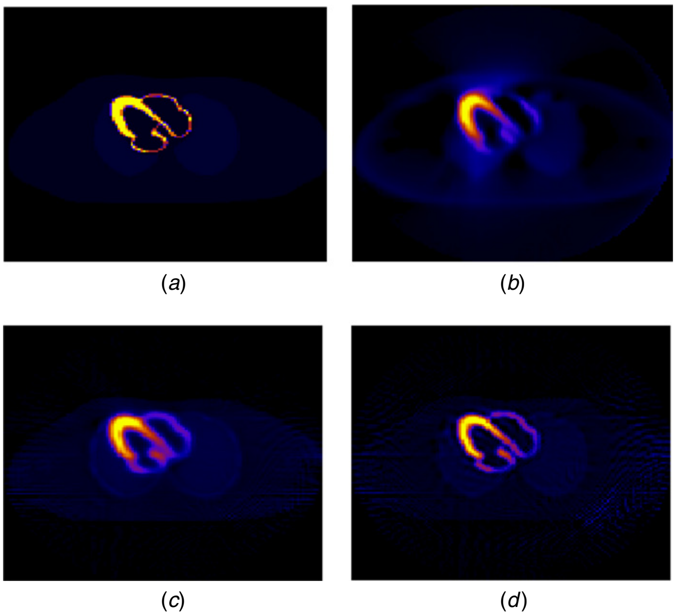

Figure 2. Different reconstructions of a cardiac phantom.

in most of the hospitals). The reconstruction in (d) uses an improved mathematical model for SPECT (which takes into account the fact that the collimator actually receives 'cones' instead of rays); this leads to a modified attenuated Radon transform which can also be inverted analytically [26]. The incorporation of noise to these analytical algorithms is a difficult problem which is under consideration.

3. A nonlinear Fourier transform pair in four dimensions

We first introduce some useful notations.

- $S(\mathbb{R}^4)$ will denote the space of Schwartz functions in four dimensions.
- x, y, k denote the following complex variables:

$$x = x_1 + \mathrm{i}x_2, \qquad y = y_1 + \mathrm{i}y_2, \qquad k = k_1 + \mathrm{i}k_2, \qquad (x_1, x_2, y_1, y_2, k_1, k_2) \in \mathbb{R}^6. \tag{3.1}$$

- $f(x, y, k)$ denotes $f(x_1, x_2, y_1, y_2, k_1, k_2)$
- If χ is the complex variable $\chi = \chi_1 + \mathrm{i}\chi_2$, $(\chi_1, \chi_2) \in \mathbb{R}^2$, then $\mathrm{d}\chi = \mathrm{d}\chi_1 \mathrm{d}\chi_2$.

Proposition 3.1. *Assume that the complex-valued function $f(x_1, x_2, y_1, y_2) \in S(\mathbb{R}^4)$ satisfies appropriate 'small norm' conditions such that the following linear integral equation has a unique solution in the Banach space of bounded continuous complex-valued functions in $\mathbb{R}^6$:*

$$\mu(x, y, k) = 1 + \int_{\mathbb{R}^4} G(x - x', y - y', k) f(x', y') \mu(x', y', k)\, \mathrm{d}x'\, \mathrm{d}y', \tag{3.2}$$

where the function G is defined by

$$G(x, y, k) = -\frac{1}{\pi^4} \int_{\mathbb{R}^4} \frac{\mathrm{e}^{\xi x - \bar{\xi}\bar{x} + \eta y - \bar{\eta}\bar{y}}}{-\bar{\eta} - \bar{\xi}^2 + 2k\bar{\xi}}\, \mathrm{d}\xi\, \mathrm{d}\eta, \qquad \xi = \xi_1 + \mathrm{i}\xi_2, \qquad \eta = \eta_1 + \mathrm{i}\eta_2,$$
$$(x_1, x_2, k_1, k_2, \xi_1, \xi_2, \eta_1, \eta_2) \in \mathbb{R}^8. \tag{3.3}$$

Then, the following nonlinear Fourier transform of f is well defined

$$\mathbb{R}^4 \to \mathbb{R}^4 : f(x_1, x_2, y_1, y_2) \to \hat{f}(k_1, k_2, \lambda_1, \lambda_2)$$
$$\hat{f}(k,\lambda) = \frac{2}{\pi^3}\int_{\mathbb{R}^4} (\bar{\lambda} - \bar{k})\bar{E}(k,\lambda,x,y) f(x,y)\mu(x,y,\lambda\, \mathrm{d}x\, \mathrm{d}y, \tag{3.4}$$

where μ is uniquely defined in terms of f by equation (3.2) and E denotes the exponential

$$E(k,\lambda,x,y) = \mathrm{e}^{2\mathrm{i}[(\lambda_2-k_2)x_1+(k_1-\lambda_1)x_2+2(\lambda_1\lambda_2-k_1k_2)y_1+(k_1^2-k_2^2+\lambda_2^2-\lambda_1^2)y_2]}. \tag{3.5}$$

Assume that the complex-valued function $\hat{f}(k,\lambda) \in S(\mathbb{R}^4)$ satisfies appropriate 'small norm' conditions such that the following linear integral equation has a unique solution in the Banach space of bounded continuous complex-valued functions in $\mathbb{R}^6$,

$$\mu(x,y,k) = 1 + \frac{1}{\pi}\int_{\mathbb{R}^4} E(k',\lambda,x,y)\hat{f}(k',\lambda)\mu(x,y,\lambda)\frac{\mathrm{d}k'\,\mathrm{d}\lambda}{k-k'}, \qquad k \in \mathbb{C}. \tag{3.6}$$

Then, the inverse nonlinear Fourier transform associated with equation (3.4),

$$\mathbb{R}^4 \to \mathbb{R}^4 : \hat{f}(k_1, k_2, \lambda_1, \lambda_2) \to f(x_1, x_2, y_1, y_2)$$

is given by the equation

$$f(x,y) = \frac{2}{\pi}\partial_{\bar{x}}\int_{\mathbb{R}^4} E(k,\lambda,x,y)\hat{f}(k,\lambda)\mu(x,y,\lambda)\,\mathrm{d}k\,\mathrm{d}\lambda, \tag{3.7}$$

where μ is uniquely defined in terms of $\hat{f}$ by equation (3.6).

Proof. We will derive the above nonlinear transform pair by performing the spectral analysis of the eigenvalue equation

$$\mu_{\bar{y}} - \mu_{\bar{x}\bar{x}} - 2k\mu_{\bar{x}} + f\mu = 0, \tag{3.8}$$

where the complex variables (x, y, k) are defined in (3.1) and $f(x,y) \in S(\mathbb{R}^4)$.

In order to solve the direct problem we look for a solution of equation (3.8) such that

$$\mu \to 1 \quad \text{as} \quad |x|^2 + |y|^2 \to \infty. \tag{3.9}$$

A solution of equation (3.8) satisfying equation (3.9) is given by equation (3.2). Indeed,

$$\mu(x,y,k) = 1 + \int G(x-x', y-y', k) f(x',y')\mu(x',y',k)\,\mathrm{d}x'\,\mathrm{d}y', \tag{3.10}$$

where the Green's function $G(x, y, k)$ satisfies

$$G_{\bar{y}} - G_{\bar{x}\bar{x}} - 2kG_{\bar{x}} = -\delta(x)\delta(y). \tag{3.11}$$

Using the formula

$$\delta(x_1) = \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}\xi_1 x_1}\,\mathrm{d}\xi_1, \qquad x_1 \in \mathbb{R},$$

it follows that

$$\delta(x)\delta(y) = \frac{1}{\pi^4}\int_{\mathbb{R}^4} \mathrm{e}^{2\mathrm{i}(\xi_1 x_2+\xi_2 x_1+\eta_1 y_2+\eta_2 y_1)}\,\mathrm{d}\xi\,\mathrm{d}\eta = \frac{1}{\pi^4}\int_{\mathbb{R}^4} \mathrm{e}^{\xi x - \bar{\xi}\bar{x} + \eta y - \bar{\eta}\bar{y}}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{3.12}$$

and then equation (3.11) implies equation (3.3).

If the linear integral equation (3.2) has a unique solution then μ is bounded for all $k \in \mathbb{C}$.

In order to solve the inverse problem we must express $\partial\mu/\partial\bar{k}$ in terms of $\hat{f}$. In this respect we note that the function $\partial\mu/\partial\bar{k}$ satisfies the following linear integral equation:

$$\frac{\partial\mu}{\partial\bar{k}}(x,y,k) = \int_{\mathbb{R}^2} E(k,\lambda,x,y)\hat{f}(k,\lambda)\,\mathrm{d}\lambda + \int_{\mathbb{R}^4} G(x-x', y-y', k) f(x',y')\frac{\partial\mu}{\partial\bar{k}}(x',y',k). \tag{3.13}$$

Indeed, computing the *d*-*k*bar derivative of equation (3.2) we find an equation similar to equation (3.2) but with the first term of the rhs of equation (3.2) replaced by

$$\int_{\mathbb{R}^4} \frac{\partial G}{\partial \bar{k}}(x-x', y-y', k) f(x', y')\mu(x', y', k)\, \mathrm{d}x'\, \mathrm{d}y'. \tag{3.14}$$

We will now show that this term equals the first term of the rhs of equation (3.13). Using the identity

$$-\bar{\eta} - \bar{\xi}^2 + 2k\bar{\xi} = \left[2(k_1\xi_1 + k_2\xi_2) - \eta_1 - \xi_1^2 + \xi_2^2\right] + \mathrm{i}[\eta_2 + 2\xi_1\xi_2 + 2(k_2\xi_1 - k_1\xi_2)], \tag{3.15}$$

it follows (see appendix A) that

$$\begin{aligned}\frac{\partial G}{\partial \bar{k}} &= -\frac{2}{\pi^3}\int_{\mathbb{R}^2} \mathrm{e}^{2\mathrm{i}(\xi_1 x_2 + \xi_2 x_2 + \eta_1 y_2 + \eta_2 y_1)}\xi\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2 \bigg|_{\substack{\eta_1 = 2(k_1\xi_1 + k_2\xi_2) - \xi_1^2 + \xi_2^2 \\ \eta_2 = 2(k_1\xi_2 - k_2\xi_1) - 2\xi_1\xi_2}} \\ &= -\frac{2}{\pi^3}\int_{\mathbb{R}^2} \exp\left\{2\mathrm{i}[\xi_1 x_2 + \xi_2 x_1 + 2(k_1\xi_1 + k_2\xi_2)y_2 + \left(\xi_2^2 - \xi_1^2\right)y_2\right. \\ &\quad \left. + \, 2(k_1\xi_2 - k_2\xi_1)y_1 - 2\xi_1\xi_2 y_1]\right\}\xi\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2.\end{aligned}$$

The change of variables

$$\xi_1 = -\lambda_1 + k_1, \qquad \xi_2 = \lambda_2 - k_2$$

implies

$$\frac{\partial G}{\partial \bar{k}} = \frac{2}{\pi^3}\int_{\mathbb{R}^2} (\bar{\lambda} - \bar{k})E(k, \lambda, x, y)\, \mathrm{d}\lambda,$$

hence, the expression in (3.15) becomes the first term of the rhs of equation (3.13).

Multiplying equation (3.2) with k replaced by λ by $E(k, \lambda, x, y)\hat{f}(k, \lambda)$, integrating over $\mathrm{d}\lambda$ and comparing the resulting equation with equation (3.13) we find

$$\frac{\partial \mu}{\partial \bar{k}} = \int_{\mathbb{R}^2} E(k, \lambda, x, y)\hat{f}(k, \lambda)\mu(x, y, \lambda)\, \mathrm{d}\lambda, \tag{3.16}$$

provided that G satisfies the symmetry relation

$$G(x, y, \lambda)E(k, \lambda, x, y) = G(x, y, k).$$

This identity can be verified by replacing in equation (3.3) the variables $(\xi_1, \xi_2, \eta_1, \eta_2)$ with

$$\left(\xi_1 + k_1 - \lambda_1, \xi_2 + \lambda_2 - k_2, \eta_1 + k_1^2 - k_2^2 + \lambda_2^2 - \lambda_1^2, \eta_2 + 2(\lambda_1\lambda_2 - k_1k_2)\right).$$

Integrating equation (3.16) and using the estimate that

$$\mu \to 1 \quad \text{as} \quad k \to \infty,$$

we find equation (3.6).

Equations (3.2) and (3.6) express μ in terms of f and $\hat{f}$ respectively. The easiest way to obtain a relation between f and $\hat{f}$ is to note that as $k \to \infty$ equations (3.6) and (3.8) imply respectively

$$\mu \sim 1 + \frac{1}{\pi k}\int_{\mathbb{R}^4} E(k, \lambda, x, y)\hat{f}(k, \lambda)\mu(x, y, \lambda)\, \mathrm{d}k\, \mathrm{d}\lambda + O\left(\frac{1}{k^2}\right),$$

$$\mu \sim 1 + \frac{1}{2k}\partial_{\bar{x}}^{-1} f + O\left(\frac{1}{k^2}\right).$$

The $O\left(\frac{1}{k}\right)$ terms of the above equations yield equations (3.7). □

Remark 3.1. In the linear limit of small f, equations (3.4) and (3.7) can be obtained from the usual Fourier transform pair after a change of variables, see appendix B.

Remark 3.2. Using the definition (3.4) it is possible to express the small norm condition for $\hat{f}$ in terms of appropriate small norm conditions for f.

3.1. A generalization of the Kadomtsev–Petviashvilli equation in 4 + 2

Suppose that $\hat{f}(k,\lambda)$ is allowed to depend on the *complex* variable t,

$$t = t_1 + \mathrm{i}t_2, \qquad (t_1, t_2) \in \mathbb{R}^2; \tag{3.17}$$

let $\hat{q}(k,\lambda,t)$ denote this function. Then $f(x,y)$ will also depend on t and we denote this function by $q(x,y,t)$. Using the so-called dressing method [27] it can be shown that if

$$\hat{q}(k,\lambda,t) = \hat{f}(k,\lambda)\,\mathrm{e}^{(\lambda^3-k^3)\bar{t}-(\bar{\lambda}^3-\bar{k}^3)t}, \tag{3.18}$$

then $\mu(x,y,t,k)$ in addition to the equation

$$\mu_{\bar{y}} - \mu_{\bar{x}\bar{x}} - 2k\mu_{\bar{x}} + q\mu = 0, \tag{3.19}$$

also satisfies the following eigenvalue equation:

$$\mu_{\bar{t}} - \mu_{\bar{x}\bar{x}\bar{x}} - 3k^2\mu_{\bar{x}} - 3k\mu_{\bar{x}\bar{x}} + \tfrac{3}{2}q\mu_{\bar{x}} + \tfrac{3}{2}kq\mu + \tfrac{3}{4}q_{\bar{x}}\mu + \tfrac{3}{4}\big(\partial_{\bar{x}}^{-1}q_{\bar{y}}\big)\mu = 0, \tag{3.20}$$

where $\partial_{\bar{x}}^{-1}$ is defined by

$$\big(\partial_{\bar{x}}^{-1}f\big)(x) = \frac{1}{\pi}\iint_{\mathbb{R}^2} f(x')\frac{\mathrm{d}x'}{x-x'}, \qquad (x_1,x_2)\in\mathbb{R}^2. \tag{3.21}$$

Equations (3.20) and (3.21) provide a Lax pair for the following 4 + 2 generalization of the Kadomtsev–Petviashvilli equation

$$q_{\bar{t}} = \tfrac{1}{4}q_{\bar{x}\bar{x}\bar{x}} - \tfrac{3}{2}qq_{\bar{x}} + \tfrac{3}{4}\partial_{\bar{x}}^{-1}q_{\bar{y}\bar{y}}. \tag{3.22}$$

Using the nonlinear transform pair of proposition 3.1 it is straightforward to solve the Cauchy problem of equation (3.22), i.e. equation (3.22) supplemented with the initial condition

$$q(x_1,x_2,y_1,y_2,0,0) = f(x_1,x_2,y_1,y_2). \tag{3.23}$$

Indeed, suppose that the function $f \in S(\mathbb{R}^4)$ satisfies appropriate small norm conditions such that equations (3.2) and (3.6) have unique solutions (see remark 3.2). Define $\hat{f}$ in terms of f by equation (3.4) and then define $\hat{q}$ by equation (3.18). Finally, define $q(x,y,t)$ by equations (3.6) and (3.7) where $\hat{f}$ in these equations is replaced with $\hat{q}$. Then the function q satisfies equations (3.22) and (3.23).

4. A novel transform pair and the Laplace equation

Proposition 4.1. *Define $\hat{f}(k)$ in terms of $f(x)$ by*

$$\hat{f}(k) = \int_0^X \mathrm{e}^{-\mathrm{i}kx-kl(x)} f(x)\,\mathrm{d}x, \qquad k\in\mathbb{C}, \tag{4.1}$$

where $f(x)$ is an arbitrary complex-valued function with appropriate smoothness, X is a positive constant and $l(x)$ is a smooth function satisfying

$$l''(x) > 0, \qquad x\in[0,X]; \qquad l(0) = 0. \tag{4.2}$$

Then $f(x)$ can be expressed in terms of $\hat{f}(k)$ through the solution of the following linear Fredholm integral equation:

$$f(x) = \frac{1-\mathrm{i}l'(x)}{2\pi}\left\{\int_{L(x)} \mathrm{e}^{\mathrm{i}kx+kl(x)}\hat{f}(k)\,\mathrm{d}k + \int_0^X \frac{f(s)}{l(x)-l(s)+\mathrm{i}(x-s)}\mathrm{d}s\right\}, \qquad 0<x<X, \tag{4.3}$$

where the ray $L(x)$ depicted in figure 3, is defined by

$$L(x) = \{k_1 \geqslant 0, k_2 = k_1 l'(x), x\in(0,X)\}. \tag{4.4}$$

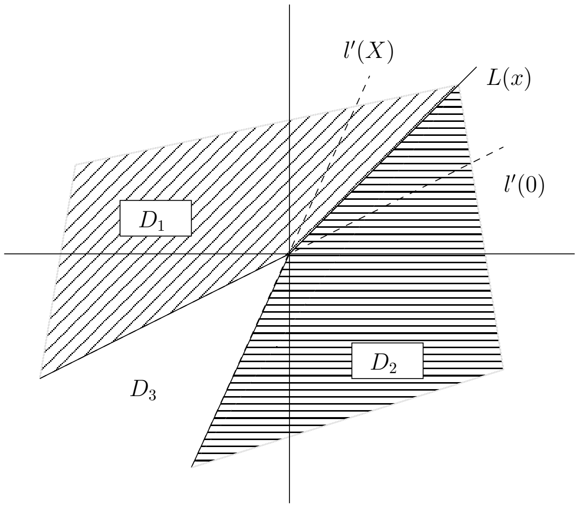

Figure 3. The domains D_j, $j = 1, 2, 3$.

Proof. We will derive the above linear transform pair by performing the spectral analysis of the eigenvalue equation

$$\frac{\partial \mu}{\partial x}(x,k) - \mathrm{i}k(1 - \mathrm{i}l'(x))\mu(x,k) = f(x), \qquad k = k_1 + \mathrm{i}k_2, \qquad x \in (0, X),$$
$$(k_1, k_2) \in \mathbb{R}^2. \tag{4.5}$$

In order to solve the direct problem we define the following x-dependent domains in the complex k-plane, see figure 3:

$$D_1(x) = \{k_1 > 0, k_2 > l'(x)\} \cup \{k_1 < 0, k_2 > l'(0)k_1\}, \tag{4.6a}$$
$$D_2(x) = \{k_1 > 0, k_2 < l'(x)k_1\} \cup \{k_1 < 0, k_2 < l'(X)k_1\}, \tag{4.6b}$$
$$D_3(x) = \mathbb{C}\backslash\{D_1(x) \cup D_2(x)\}. \tag{4.6c}$$

A solution $\mu(x,k)$ of equation (4.5) bounded for all complex values of k is given by

$$\mu(x,k) = \mu_j(x,k), \qquad k \in D_j(x), \qquad 0 < x < X, \quad j = 1, 2, 3, \tag{4.7}$$

where

$$\mu_j(x,k) = \int_{x_j}^{x} \mathrm{e}^{\mathrm{i}k(x-\xi)+k[l(x)-l(\xi)]} f(\xi)\,\mathrm{d}\xi, \qquad j = 1, 2, 3, \tag{4.8}$$

$$x_1 = 0, \qquad x_2 = X, \qquad x_3 = S\left(\frac{k_2}{k_1}\right) \tag{4.9}$$

and the function $S(k)$ is the inverse of the function $l'(x)$:

$$k = l'(x) \longleftrightarrow x = S(k), \qquad k \in D_3, \qquad x \in (0, X). \tag{4.10}$$

Indeed, the functions μ_1 and μ_2 are entire functions of k, which are bounded as $k \to \infty$ in the domains D_1 and D_2 respectively. These domains are determined by the real part of the exponential appearing in equation (4.8), which equals

$$\exp\{-(x-\xi)[k_2 - k_1 l'(\tau)]\}, \qquad \tau \in (x, \xi).$$

For μ_1, $x-\xi \geqslant 0$, thus the above exponential is bounded as $k \to \infty$ provided that $k_2 \geqslant k_1 l'(\tau)$, hence

$$k_1 \geqslant 0, \qquad \frac{k_2}{k_1} \geqslant l'(\tau); \qquad k_1 \leqslant 0, \qquad \frac{k_2}{k_1} \leqslant l'(\tau).$$

Taking into consideration that $l'(\tau)$ is an increasing function, the inequalities involving $l'(\tau)$ are satisfied provided that

$$k_1 \geqslant 0, \qquad \frac{k_2}{k_1} \geqslant l'(x); \qquad k_1 \leqslant 0, \qquad \frac{k_2}{k_1} \leqslant l'(0). \tag{4.11a}$$

Similarly for μ_2, $x - \xi \leqslant 0$, thus boundness requires

$$k_1 \geqslant 0, \qquad \frac{k_2}{k_1} \leqslant l'(\tau); \qquad k_1 \leqslant 0, \qquad \frac{k_2}{k_1} \geqslant l'(\tau),$$

which yields

$$k_1 \geqslant 0, \qquad \frac{k_2}{k_1} \leqslant l'(x), \qquad k_1 \leqslant 0, \qquad \frac{k_2}{k_1} \geqslant l'(X). \tag{4.11b}$$

Equations (4.11*a*) and (4.11*b*) imply the definitions (4.6*a*) and (4.6*b*).

The domains $D_1 \cup D_2$ do *not* cover the entire complex k-plane, thus it is necessary to introduce the function μ_3. It is straightforward to show that μ_3 is bounded in D_3 but since S depends on k_2/k_1, μ_3 is *not* an analytic function.

The definitions of μ_j imply that μ_3 coincides with the functions μ_1 and μ_2 on the rays $\{k_1 < 0, k_2 = l'(0)k_1\}$ and $\{k_1 < 0, k_2 = l'(X)k_1\}$ respectively. Furthermore, $\mu_j = 0(1/k)$ as $k \to \infty$. Hence the Pompieu (or d-bar, or Cauchy–Green) formula [28] implies

$$\mu(x,k) = \frac{1}{2\mathrm{i}\pi}\int_{L(x)} (\mu_1 - \mu_2)(x,\lambda)\frac{\mathrm{d}\lambda}{\lambda - k} + \frac{1}{\pi}\iint_{D_3} \frac{\partial \mu_3}{\partial \bar{\lambda}}(x,\lambda)\frac{\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2}{k\lambda},$$
$$\lambda = \lambda_1 + \mathrm{i}\lambda_2, \qquad (\lambda_1,\lambda_2) \in \mathbb{R}^2, \qquad k \in \mathbb{C}\backslash L(x). \tag{4.12}$$

The definitions of $\{\mu_j\}_1^3$ imply

$$\mu_1(x,k) - \mu_2(x,k) = \mathrm{e}^{\mathrm{i}kx + kl(x)}\hat{f}(k), \tag{4.13a}$$

$$\frac{\partial \mu_3}{\partial \bar{k}} = \frac{\mathrm{i}\bar{k}}{2k_1^2}\frac{1}{l''(S)}\mathrm{e}^{\mathrm{i}k(x-S)+k[l(x)-l(S)]}q(S), \tag{4.13b}$$

where we have used the formula

$$\frac{\partial}{\partial \bar{k}} = \frac{1}{2}\left(\frac{\partial}{\partial k_1} + \mathrm{i}\frac{\partial}{\partial k_2}\right).$$

Equations (4.7), where $\{\mu_j\}_1^3$ are defined by equations (4.8) and (4.9), provide the solution of the direct problem. Equation (4.12), where $\mu_1 - \mu_2$ and $\partial\mu_3/\partial\bar{k}$ are given by equations (4.13), provides the solution of the inverse problem. Substituting equation (4.12) into equation (4.5) we find

$$f(x) = \frac{1 - \mathrm{i}l'(x)}{2\pi}\left\{\int_{L(x)} \mathrm{e}^{\mathrm{i}kx + kl(x)}\hat{f}(k)\,\mathrm{d}k - \iint_D \frac{\bar{k}\,\mathrm{e}^{\mathrm{i}k(x-S)+k[l(x)-l(S)]}q(S)}{k_1^2 l''(S)}\,\mathrm{d}k_1\,\mathrm{d}k_2\right\}. \tag{4.14}$$

The second term of the rhs of equation (4.14) can be transformed to the second term of the rhs of equation (4.3) by using the transformations

$$\frac{k_2}{k_1} = \lambda, \qquad k_2 = \nu, \qquad (\lambda, \nu) \in \mathbb{R}^2$$

and

$$S(\lambda) = s \leftrightarrow \lambda = l'(s), \qquad (s, \lambda) \in \mathbb{R}^2.$$

These transformations imply

$$k = (1 + \mathrm{i}\lambda)\nu, \qquad \mathrm{d}k_1\,\mathrm{d}k_2 = \nu\mathrm{d}\nu\,\mathrm{d}\lambda; \qquad \mathrm{d}\lambda = l''(s)\,\mathrm{d}s.$$

Under these transformations the second term of the rhs of equation (4.14) yields

$$\int_{-\infty}^{0} \mathrm{d}\nu \int_{l'(X)}^{l'(0)} \mathrm{d}\lambda \frac{(1 + \mathrm{i}\lambda)}{l''(S)}\, \mathrm{e}^{[\mathrm{i}(x-S)+l(x)-l(S)]\nu(1+\mathrm{i}\lambda)} q(S) = \int_X^0 [1 + \mathrm{i}l'(s)] K(x, s) q(s)\,\mathrm{d}s, \quad (4.15)$$

where

$$K(x, s) = \int_{-\infty}^{0} \exp\{\nu[1 + \mathrm{i}l'(s)][\mathrm{i}(x - s) + l(x) - l(s)]\}\,\mathrm{d}\nu.$$

The real part of the product of the two brackets appearing in the above exponential is positive, thus K is well defined. Computing K explicitly, we find that the rhs of equation (4.15) yields the rhs of equation (4.3). □

Remark 4.1. The representation (4.12) was first derived in [29]. However, it was not realized in [29] that the double integral appearing in equation (4.12) can be expressed as a Fredholm integral term. The approach followed here was developed in [17].

Remark 4.2. The integral defined in (4.1) appears in the characterization of the Dirichlet-to-Neumann map of the Laplace equation in the interval of a domain which involves the curve $l(x)$. Indeed, the Laplace equation can be written in the form

$$\frac{\partial}{\partial \bar{z}} q_z = 0, \qquad z = x + \mathrm{i}y, \qquad (x, y) \in \mathbb{R}^2,$$

which implies that q_z is an analytic function. Hence, if q satisfies the Laplace equaiton in a piecewise smooth domain $\Omega \subset \mathbb{R}^2$ then

$$\int_{\partial\Omega} \mathrm{e}^{-\mathrm{i}kz} q_z \mathrm{d}z = 0, \qquad k = k_1 + \mathrm{i}k_2, \qquad (k_1, k_2) \in \mathbb{R}^2,$$

where $\partial\Omega$ denotes the boundary of Ω. If part of $\partial\Omega$ is given by $\{0 < x < X, y = l(x)\}$, then this part of the boundary yields the following term:

$$\frac{1}{2} \int_0^X \mathrm{e}^{-\mathrm{i}kx + kl(x)} [q_x(x, l(x)) - \mathrm{i}q_y(x, l(x))](1 + \mathrm{i}l'(x))\,\mathrm{d}x.$$

For the Dirichlet problem q is given, thus

$$q(x, l(x)) = d(x), \qquad 0 < x < X,$$

where $d(x)$ is a given complex-valued function. Thus,

$$q_x(x, l(x)) - \mathrm{i}q_y(x, l(x)) = [1 + \mathrm{i}l'(x)]d'(x) - \mathrm{i}[1 + l'(x)^2]q_y(x, l(x)).$$

Hence the above integral becomes

$$\int_0^X \mathrm{e}^{-\mathrm{i}kx + kl(x)} f(x)\,\mathrm{d}x + \hat{f}(k),$$

where $f(x)$ is an unknown function, whereas $\hat{f}(k)$ is known:

$$f(x) = -\mathrm{i}[1 + l'^2(x)]q_y(x, l(x)), \qquad \hat{f}(k) = \int_0^X \mathrm{e}^{-\mathrm{i}kx + kl(x)}[1 + \mathrm{i}l'(x)]d'(x).$$

5. Conclusions

The spectral analysis of a given eigenvalue equation and the formulation of a d-bar problem provides a powerful tool for the construction of linear and nonlinear transform pairs. Regarding equation (3.8) we note that although μ depends on *four* spatial variables, (x_1, x_2, y_1, y_2), μ only depends on *two* spectral variables, (k_1, k_2). It is interesting that the additional *two* variables (λ_1, λ_2) needed for the formulation of the nonlinear Fourier transform are generated by the *nonlocal* d-bar problem. This mechanism is similar to that in the Kadomtsev–Petviashvilli I equation, where the corresponding μ depends on *two* spatial variables but only one spectral variable k_1; in that case, the second spectral variable k_2 is generated by the *nonlocal* Riemann–Hilbert problem.

It is possible to reduce equation (3.23) from 4 + 2 to 3 + 1 dimensions, but this reduction is *implicit*: the spectral function $\hat{q}$ is independent of t_1 provided that

$$\operatorname{Im}(\lambda^3 - k^3) = 0, \qquad \text{or} \qquad k_2^3 - \lambda_2^3 + 3\lambda_1^2\lambda_2 - 3k_1^2 k_2 = 0. \tag{5.1}$$

This constraint in the Fourier space implies a constraint in the physical space and hence the constraint (5.1) reduces equation (3.22) to an equation in 3 + 1. For small q, this reduction is explicit, however the question of writing the relevant constraint in the physical space explicitly when q is *not* small, remains open.

An alternative reduction, which is more convenient for the potential version of equation (3.22), involves $\partial_{y_2} = -\partial_{x_1}^{-1}\partial_{x_2}\partial_{y_1}$, as well as

$$\left(\partial_{x_1}^2 + \partial_{x_2}^2\right)q = 0. \tag{5.2}$$

In this case q satisfies [21]

$$q_{t_1} = \tfrac{1}{4}q_{x_1x_1x_1} - \tfrac{3}{8}\left(q_{x_1}^2 - q_{x_2}^2\right) + \tfrac{3}{4}\partial_{x_1}^{-1}q_{y_1y_1}. \tag{5.3}$$

It can be shown that the evolution defined by equation (5.3) preserves the Laplace equation (5.2). Thus, equation (5.3) is an integrable nonlinear evolution equation in 3 + 1, namely x_1, x_2, y_1 and t_1, provided that $q(x_1, x_2, y, 0)$ is harmonic in x_1 and x_2.

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Appendix A.

In what follows we will show that if G is defined by

$$G(x, y, k) = -\frac{1}{\pi^4}\int_{\mathbb{R}^4} \frac{e^{\xi x - \bar{\xi}\bar{x} + \eta y - \bar{\eta}\bar{y}}}{-\bar{\eta} - \bar{\xi}^2 + 2k\bar{\xi}}\,d\xi\,d\eta,$$

then

$$\frac{\partial G}{\partial \bar{k}}(x, y, k) = -\frac{2}{\pi^3}\int_{\mathbb{R}^2} e^{\xi x - \bar{\xi}\bar{x} + \eta y - \bar{\eta}\bar{y}}\xi\,d\xi\,\Bigg|_{\substack{\eta_1 = 2(k_1\xi_1 + k_2\xi_2) - \xi_1^2 + \xi_2^2 \\ \eta_2 = 2(k_1\xi_2 - k_2\xi_1) - 2\xi_1\xi_2}}. \tag{A.1}$$

Indeed,

$$\frac{\partial G}{\partial \bar{k}} = -\frac{1}{\pi^4}\int_{\mathbb{R}^4} \frac{e^{\xi x - \bar{\xi}\bar{x} + \eta y - \bar{\eta}\bar{y}}}{2\bar{\xi}}\frac{\partial}{\partial \bar{k}}\left(\frac{1}{\frac{-\bar{\eta} - \bar{\xi}^2}{2\bar{\xi}} + k}\right)d\xi\,d\eta.$$

But

$$\frac{\bar{\eta}+\bar{\xi}^2}{2\bar{\xi}} = \frac{1}{2|\xi|^2}(\eta_1\xi_1+\eta_2\xi_2)+\xi_1/2+\mathrm{i}\left(\frac{1}{2|\xi|^2}(\eta_1\xi_2-\eta_2\xi_1)-\xi_2/2\right).$$

Hence

$$\frac{\partial}{\partial\bar{k}}\left(\frac{1}{\frac{-\bar{\eta}-\bar{\xi}^2}{2\bar{\xi}}+k}\right) = \pi\delta\left(k_1-\mathrm{Re}\left(\frac{\bar{\eta}+\bar{\xi}^2}{2\bar{\xi}}\right)\right)\delta\left(k_2-\mathrm{Im}\left(\frac{\bar{\eta}+\bar{\xi}^2}{2\bar{\xi}}\right)\right)$$
$$= \pi\delta\left(k_1-\frac{1}{2|\xi|^2}(\eta_1\xi_1+\eta_2\xi_2)-\xi_1/2\right)\delta\left(k_2-\frac{1}{2|\xi|^2}(\eta_1\xi_2-\eta_2\xi_1)+\xi_2/2\right).$$

We can use the first delta function to compute the integral with respect to $\mathrm{d}\eta_2$ (note that η_2 is multiplied by $\frac{\xi_2}{2|\xi|^2}$ in the delta function). Thus

$$\frac{\partial G}{\partial\bar{k}} = -\frac{1}{\pi^4}\int_{\mathbb{R}^3}\frac{\mathrm{e}^{\xi x-\bar{\xi}\bar{x}+\eta y-\bar{\eta}\bar{y}}}{2\bar{\xi}}\pi\left|\frac{2|\xi|^2}{\xi_2}\right|$$
$$\times\,\delta\left(k_2-\frac{1}{2|\xi|^2}(\eta_1\xi_2-\eta_2\xi_1)+\xi_2/2\right)\Bigg|_{\eta_2=\frac{1}{\xi_2}((k_1-\xi_1/2)2|\xi|^2-\eta_1\xi_1)}\mathrm{d}\xi\,\mathrm{d}\eta_1.$$

Computing the integral with respect to $\mathrm{d}\eta_1$ (note that η_1 is multiplied by $\frac{1}{2|\xi|^2}(\xi_1^2/\xi_2+\xi_2) = 1/(2\xi_2)$ in the delta function), we find

$$-\frac{1}{\pi^4}\int_{\mathbb{R}^2}\frac{\mathrm{e}^{\xi x-\bar{\xi}\bar{x}+\eta y-\bar{\eta}\bar{y}}}{2\bar{\xi}}\pi\left|\frac{2|\xi|^2}{\xi_2}\right||2\xi_2|\,\mathrm{d}\xi\,\Bigg|_{\substack{\eta_1=2(k_1\xi_1+k_2\xi_2)-\xi_1^2+\xi_2^2\\ \eta_2=2(k_1\xi_2-k_2\xi_1)-2\xi_1\xi_2}},$$

which simplifies to equation (A.1).

Appendix B. (The linear limit of the Fourier transform pair)

In the linear limit $f\to\epsilon f+O(\epsilon^2)$, we find $\mu = 1+O(\epsilon)$ so that to first order the Fourier transform pair (3.4) and (3.7) becomes

$$\hat{f}(k,\lambda) = \frac{2}{\pi^3}\int_{\mathbb{R}^4}(\bar{\lambda}-\bar{k})\bar{E}(k,\lambda,x,y)f(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{B.1}$$

$$f(x,y) = \frac{2}{\pi}\partial_{\bar{x}}\int_{\mathbb{R}^4}E(k,\lambda,x,y)\hat{f}(k,\lambda)\,\mathrm{d}k\,\mathrm{d}\lambda. \tag{B.2}$$

Using

$$\partial_{\bar{x}}E(k,\lambda,x,y) = \tfrac{1}{2}(\partial_{x_1}+\mathrm{i}\partial_{x_2})E(k,\lambda,x,y) = (\lambda-k)E(k,\lambda,x,y),$$

equation (B.2) becomes

$$f(x,y) = \frac{2}{\pi}\int_{\mathbb{R}^4}(\lambda-k)E(k,\lambda,x,y)\hat{f}(k,\lambda)\,\mathrm{d}k\,\mathrm{d}\lambda. \tag{B.3}$$

We will show that equations (B.1) and (B.3) can be obtained from the usual Fourier transform pair after a change of variables. Indeed, the Fourier transform is defined by

$$\hat{\varphi}(\mathbf{p}) = \int\mathrm{d}^n x\,\mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}f(\mathbf{x}),$$

$$\varphi(\mathbf{x}) = \frac{1}{(2\pi)^n}\int\mathrm{d}^n p\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\hat{f}(\mathbf{p}),$$

where $\mathbf{x}$ and $\mathbf{p}$ are n-dimensional vectors. Letting

$$\mathbf{p} = \left(2(\lambda_2 - k_2), 2(k_1 - \lambda_1), 4(\lambda_1\lambda_2 - k_1k_2), 2\left(k_1^2 - k_1^2 + \lambda_2^2 - \lambda_1^2\right)\right)$$

and

$$\mathbf{x} = (x_1, x_2, y_1, y_2),$$

we find

$$E(k, \lambda, x, y) = \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}.$$

Furthermore, letting $\varphi(\mathbf{x}) = f(x, y)$, equation (B.1) yields

$$\begin{aligned}\hat{f}(k, \lambda) &= \frac{2}{\pi^3}\int_{\mathbb{R}^4}(\bar{\lambda} - \bar{k})\bar{E}(k, \lambda, x, y) f(x, y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{2}{\pi^3}(\bar{\lambda} - \bar{k})\int_{\mathbb{R}^4}\mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\varphi(\mathbf{x})\,\mathrm{d}^4x = \frac{2}{\pi^3}(\bar{\lambda} - \bar{k})\hat{\varphi}(\mathbf{p}).\end{aligned} \tag{B.4}$$

We must now show that $f(x, y)$ is given by (B.3). Using (B.4) on the right-hand side of equation (B.3) we find

$$\frac{2}{\pi}\int_{\mathbb{R}^4}(\lambda - k)E(k, \lambda, x, y)\hat{f}(k, \lambda)\,\mathrm{d}k\,\mathrm{d}\lambda = \frac{2}{\pi}\int_{\mathbb{R}^4}(\lambda - k)\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\frac{2}{\pi^3}(\bar{\lambda} - \bar{k})\hat{\varphi}(\mathbf{p})\,\mathrm{d}k\,\mathrm{d}\lambda. \tag{B.5}$$

In order to change variables from $\mathrm{d}k\mathrm{d}\lambda$ to d^4x we compute the determinant

$$\det\left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}k\,\mathrm{d}\lambda}\right) = \det\begin{pmatrix} 0 & -2 & 0 & 2 \\ 2 & 0 & -2 & 0 \\ -4k_2 & -4k_1 & 4\lambda_2 & 4\lambda_1 \\ 4k_1 & -4k_2 & -4\lambda_1 & 4\lambda_2 \end{pmatrix} = 64|\lambda - k|^2.$$

Therefore the integral on the right-hand side of (B.5) becomes

$$\begin{aligned}&\frac{2}{\pi}\int_{\mathbb{R}^4}(\lambda - k)\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\frac{2}{\pi^3}(\bar{\lambda} - \bar{k})\hat{\varphi}(\mathbf{p})\frac{1}{64|\lambda - k|^2}\,\mathrm{d}^4x \\ &\qquad = \frac{1}{16\pi^4}\int_{\mathbb{R}^4}\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\hat{\varphi}(\mathbf{p})\,\mathrm{d}^4x = \varphi(\mathbf{x}) = f(x, y).\end{aligned}$$

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